An operator T on a Hilbert space such that the set $eig(T)$ of all eigenvalues of T is empty but $sp(T) \neq \emptyset$.

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The unilateral shift operator on the Hilbert space l^2 (with its standard orthonormal basis (e_n)) given by $Te_n = e_{n+1}, n \in \mathcal{N}$, has no eigenvalue; since obviously $0 \notin eig(T)$ and if $0 \neq \lambda \in eig(T)$ and $Tx = \lambda x$ for some $x = \sum_{n=1}^{\infty} \alpha_n e_n \neq 0$, then $\sum_{n=1}^{\infty} \alpha_n e_{n+1} = \sum_{n=1}^{\infty} \lambda \alpha_n e_n$ and hence $\alpha_n = 0$ for all n, i.e. $x = 0$, a contradiction.

Next observe that $0 \in sp(T)$; otherwise T would be invertible so $T(T^{-1}(e_1)) = e_1$, but $< T(T^{-1}e_1), e_1 >= 0$ by the definition of T, that is impossible.