A sequence of nilpotent operators on H which converges with respect to the norm topology on $B(H)$ to an operator which is not topologically nilpotent.

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This example is due to Kakutani (cf. [Ric, p. 282]). Let H be a separable Hilbert space with orthonormal basis $(f_m)_{m\in\mathcal{N}}$. Define $\alpha_m = e^{-k}$ for $m = 2^k(2l+1),\ k,l = 0,1,\cdots$ and also the operator T by $Tf_m = \alpha_m f_{m+1}, m \in \mathcal{N}$. Then $\| T \| = \sup_{m\in\mathcal{N}} |\alpha_m|, T^n f_m = \alpha_m\alpha_{m+1}\cdots\alpha_{m+n-1}f_{m+n}$ and so $\| T^n \| = \sup_{m\in\mathcal{N}}(\alpha_m\alpha_{m+1}\cdots\alpha_{m+n-1})$.

Moreover, by the definition of the α_m, we have $\alpha_1\alpha_2\cdots\alpha_{2^t-1} = \prod_{j=1}^{t-1} exp(-j2^{t-j-1})$.

Therefore $(\alpha_1\alpha_2\cdots\alpha_{2^t-1})^{\frac{1}{2^{t-1}}} > (\prod_{j=1}^{t-1} exp[\frac{-j}{2^{j+1}}])^2$ and if $\sigma = \sum_{j=1}^{\infty}\frac{j}{2^{j+1}}$, then $e^{-2\sigma} \leq \lim_n \| T^n \|^{\frac{1}{n}}$. So T is not topologically nilpotent.

Next define the operator T_k by

$$T_k f_m = \begin{cases} 0 & m = 2^k(2l+1), l = 0,1,\cdots \\ \alpha_m f_{m+1} & \text{otherwise} \end{cases}$$

Then T_k is nilpotent. But

$$(T - T_k)f_m = \begin{cases} e^{-k} f_{m+1} & m = 2^k(2l+1), l = 0,1,\cdots \\ 0 & \text{otherwise} \end{cases}$$

Thus $\| T_k - T \| = e^{-k}$, hence $\lim_k T_k = T$ in the norm topology on $B(H)$.

Ref.

[**Ric**] C.E. Rickart, General theory of Banach algebras, Princeton, Van Nastrand, 1960.