A bounded operator on a Hilbert space which has no square root.

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Suppose that T is the operator $T(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$ on l^2 (in fact T is the adjoint of the unilateral shift operator). If T has a square root S, then $S^2 = T$ and $KerS \subseteq KerT = \mathcal{C}\xi_1$ in which $\xi_1 = (1, 0, 0, \cdots)$. Since T is not one to one we conclude that S is not one to one. So that $KerS = \mathcal{C}\xi_1$. T is surjective, hence S is onto. So there exists an element η such that $S\eta = \xi_1$. Since $T\eta = S^2\eta = S\xi_1 = 0$, we have $\eta = \lambda\xi_1$ for some $\lambda \in \mathcal{C}$ and hence $\xi_1 = S\eta = \lambda S\xi_1 = 0$, a contradiction.

Comment. There is an open subset of $L(H)$ consisting of invertible operators with no square roots.

Ref.

J.B. Conway and B.B. Morrel, Roots and logarithms of bounded operators on Hilbert spaces, J. Funct Anal, 70(1987), 171-193.