To determine the stability of the origin of the system in the new coordinate system, let us apply the Liapunov stability equation given by Equation (5–90):

$$\begin{bmatrix} 2 & 0 \\ 0.5 & 0.8 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 2 & 0.5 \\ 0 & 0.8 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = -\begin{bmatrix} 9 & 0 \\ 0 & 0.35 \end{bmatrix}$$

where we choose $\mathbf{Q}$ to be a positive definite matrix having elements that simplify the computation involved. Solving this last equation for matrix $\mathbf{P}$, we obtain

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 5 & 10 \end{bmatrix}$$

By applying the Sylvester criterion for positive definiteness, we find that matrix $\mathbf{P}$ is not positive definite. Therefore, the origin (equilibrium state) is not stable.

The instability of the equilibrium state can, of course, be determined by the z transform approach. Let us first eliminate $\hat{x}_2$ from the state equation. Then we have

$$\hat{x}_1(k+2) - 2.8\hat{x}_1(k+1) + 1.6\hat{x}_1(k) = 0$$

The characteristic equation for the system in the z plane is

$$z^2 - 2.8z + 1.6 = 0$$

or

$$(z-2)(z-0.8) = 0$$

Hence,

$$z = 2, \qquad z = 0.8$$

Since pole $z = 2$ is located outside the unit circle in the z plane, the origin (equilibrium state) is unstable.

PROBLEMS

Problem B–5–1

Obtain a state-space representation of the following pulse-transfer-function system in the controllable canonical form.

$$\frac{Y(z)}{U(z)} = \frac{z^{-1} + 2z^{-2}}{1 + 4z^{-1} + 3z^{-2}}$$

Problem B–5–2

Obtain a state-space representation of the following pulse-transfer-function system in the observable canonical form.

$$\frac{Y(z)}{U(z)} = \frac{z^{-2} + 4z^{-3}}{1 + 6z^{-1} + 11z^{-2} + 6z^{-3}}$$

Problem B–5–3

Obtain a state-space representation of the following pulse-transfer-function system in the diagonal canonical form.

$$\frac{Y(z)}{U(z)} = \frac{1 + 6z^{-1} + 8z^{-2}}{1 + 4z^{-1} + 3z^{-2}}$$

Problem B–5–4

Obtain a state-space representation of the system described by the equation

$$y(k+2) + y(k+1) + 0.16y(k) = u(k+1) + 2u(k)$$

Problem B–5–5

Obtain the state equation and output equation for the system shown in Figure 5–11.

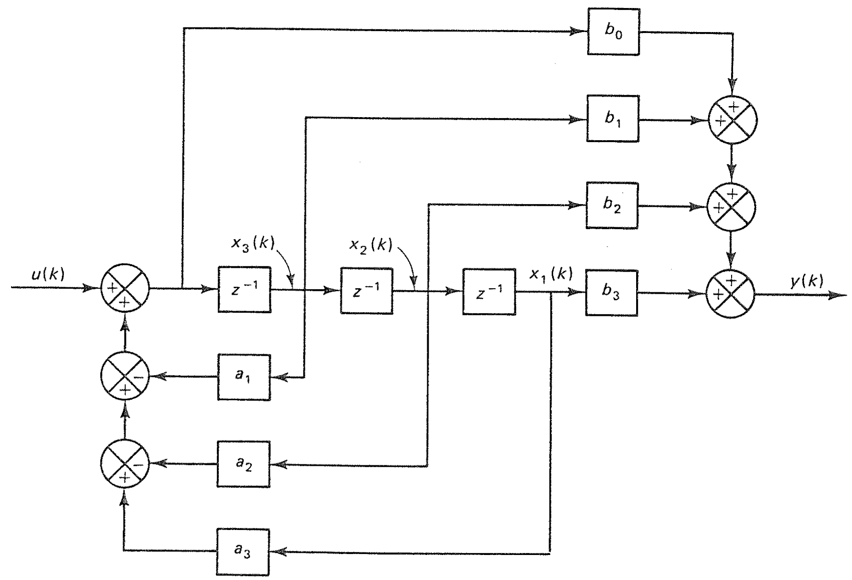

Figure 5–11 Block diagram of a control system.

Problem B–5–6

Obtain the state equation and output equation for the system shown in Figure 5–12.

Problem B–5–7

Obtain the state-space representation of the system shown in Figure 5–13.

Problem B–5–8

Figure 5–14 shows a block diagram of a discrete-time multiple-input–multiple-output system. Obtain state-space equations for the system by considering $x_1(k)$, $x_2(k)$, and $x_3(k)$ as shown in the diagram to be state variables. Then define new state variables such that the state matrix becomes a diagonal matrix.

Problem B–5–9

Obtain the state equation and output equation for the system shown in Figure 5–15.

Problem B–5–10

Obtain a state-space representation of the discrete-time control system shown in Figure 5–16.

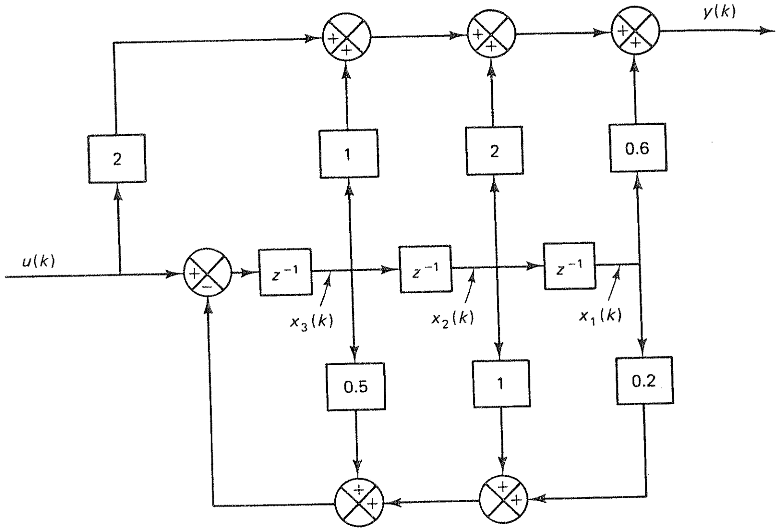

Figure 5–12 Block diagram of a control system.

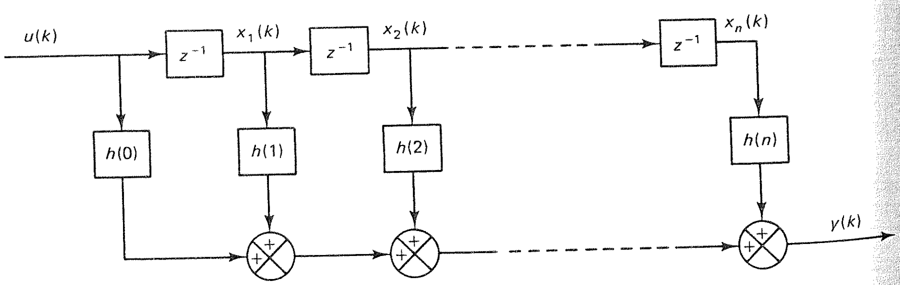

Figure 5–13 Block diagram of the control system of Problem B–5–7.

Problem B–5–11

Obtain a state-space representation of the following system in the diagonal canonical form.

$$\frac{Y(z)}{U(z)} = \frac{z^{-1} + 2z^{-2}}{1 + 0.7z^{-1} + 0.12z^{-2}}$$

Problem B–5–12

Obtain a state-space representation of the following pulse-transfer-function system such that the state matrix is a diagonal matrix:

$$\frac{Y(z)}{U(z)} = \frac{1}{(z + 1)(z + 2)(z + 3)}$$

Then obtain the initial state variables $x_1(0)$, $x_2(0)$, and $x_3(0)$ in terms of $y(0)$, $y(1)$, and $y(2)$.

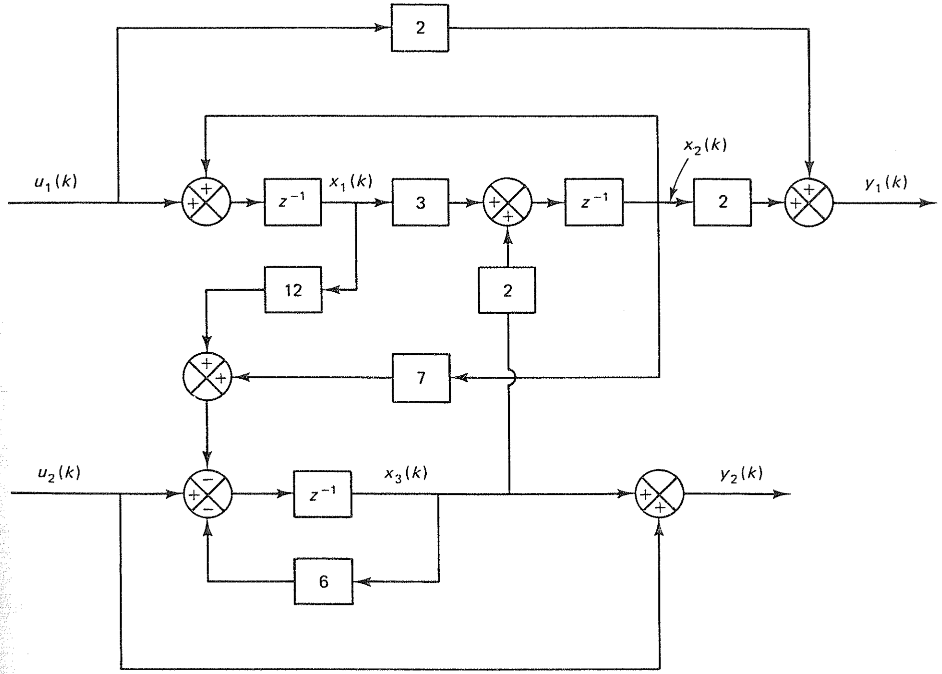

Figure 5–14 Block diagram of the discrete-time multiple-input–multiple-output system of Problem B–5–8.

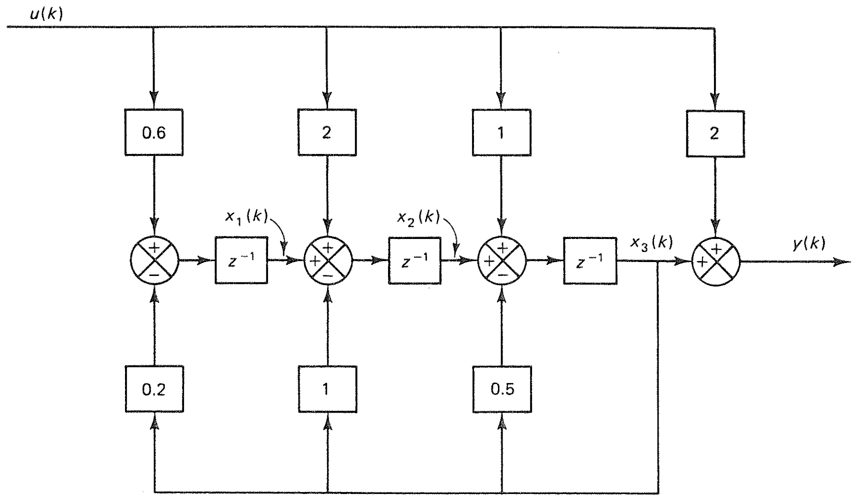

Figure 5–15 Block diagram of a control system.

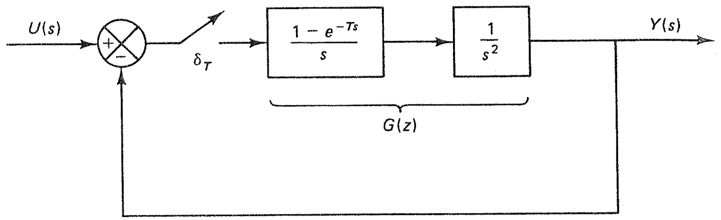

Figure 5–16 Discrete-time control system.

Problem B–5–13

A state-space representation of the scalar difference equation system

$$y(k+n) + a_1(k)y(k+n-1) + \cdots + a_n(k)y(k)$$
$$= b_0(k)u(k+n) + b_1(k)u(k+n-1) + \cdots + b_n(k)u(k)$$

where $k = 0, 1, 2, \ldots$, may be given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n(k) & -a_{n-1}(k) & \cdots & -a_2(k) & -a_1(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} h_1(k) \\ h_2(k) \\ \vdots \\ h_{n-1}(k) \\ h_n(k) \end{bmatrix} u(k)$$

$$y(k) = x_1(k) + b_0(k-n)u(k)$$

Determine $h_1(k), h_2(k), \ldots, h_n(k)$ in terms of $a_i(k)$ and $b_j(k)$, where $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, n$. Determine also the initial values of the state variables $x_1(0)$, $x_2(0), \ldots, x_n(0)$ in terms of the input sequence $u(0), u(1), \ldots, u(n-1)$ and the output sequence $y(0), y(1), \ldots, y(n-1)$.

Problem B–5–14

If the minimal polynomial of an $n \times n$ matrix $\mathbf{G}$ involves only distinct roots, then the inverse of $z\mathbf{I} - \mathbf{G}$ can be given by the following expression:

$$(z\mathbf{I} - \mathbf{G})^{-1} = \sum_{k=1}^{m} \frac{\mathbf{X}_k}{z - z_k} \qquad (5\text{–}145)$$

where m is the degree of the minimal polynomial of $\mathbf{G}$ and the $\mathbf{X}_k$'s are $n \times n$ matrices determined from

$$g_j(\mathbf{G}) = g_j(z_1)\mathbf{X}_1 + g_j(z_2)\mathbf{X}_2 + \cdots + g_j(z_m)\mathbf{X}_m$$

where

$$g_j(\mathbf{G}) = (\mathbf{G} - z_k\mathbf{I})^{j-1}, \qquad g_j(z) = (z - z_k)^{j-1}$$

where $j = 1, 2, \ldots, m$ and z_k is any one of the roots of the minimal polynomial of $\mathbf{G}$.

Using Equation (5–145), obtain $(z\mathbf{I} - \mathbf{G})^{-1}$ for the following 2×2 matrix $\mathbf{G}$:

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Problem B–5–15

Obtain the pulse transfer function of the system defined by the equations

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k) + Du(k)$$

where

$$\mathbf{G} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [b_1 - a_1 b_0 \vdots b_2 - a_2 b_0 \vdots b_3 - a_3 b_0], \qquad D = b_0$$

Problem B–5–16

Find the pulse transfer function of the system defined by

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k)$$

where

$$\mathbf{G} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0], \qquad D = b_0$$

Problem B–5–17

Obtain a state-space representation for the system defined by the following pulse-transfer-function matrix:

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1 - z^{-1}} & \dfrac{1 + z^{-1}}{1 - z^{-1}} \\ \dfrac{1}{1 + 0.6z^{-1}} & \dfrac{1 + z^{-1}}{1 + 0.6z^{-1}} \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix}$$

Problem B–5–18

Consider the discrete-time state equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.24 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Obtain the state transition matrix $\mathbf{\Psi}(k)$.

Problem B–5–19

Consider the system defined by

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

where matrix $\mathbf{G}$ is a stable matrix.

Obtain the steady-state values of $\mathbf{x}(k)$ and $\mathbf{y}(k)$ when $\mathbf{u}(k)$ is a constant vector.

Problem B–5–20

Consider the system defined by

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k)$$

where G is a stable matrix.

Show that for a positive definite (or positive semidefinite) matrix Q

$$J = \sum_{k=0}^{\infty} \mathbf{x}^*(k)\mathbf{Q}\mathbf{x}(k)$$

can be given by

$$J = \mathbf{x}^*(0)\mathbf{P}\mathbf{x}(0)$$

where $\mathbf{P} = \mathbf{Q} + \mathbf{G}^*\mathbf{P}\mathbf{G}$.

Problem B-5-21

Determine a Liapunov function $V(\mathbf{x})$ for the following system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -1.2 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Problem B-5-22

Determine the stability of the origin of the following discrete-time system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ -3 & -2 & -3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Problem B-5-23

Determine the stability of the origin of the following discrete-time system:

$$\begin{bmatrix} x_1((k+1)T) \\ x_2((k+1)T) \end{bmatrix} = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

Problem B-5-24

Consider the system defined by the equations

$$x_1(k+1) = x_1(k) + 0.2x_2(k) + 0.4$$

$$x_2(k+1) = 0.5x_1(k) - 0.5$$

Determine the stability of the equilibrium state.

6

Pole Placement and Observer Design

6-1 INTRODUCTION

In the first part of this chapter we present two fundamental concepts of control systems: controllability and observability. Controllability is concerned with the problem of whether it is possible to steer a system from a given initial state to an arbitrary state: a system is said to be controllable if it is possible by means of an unbounded control vector to transfer the system from any initial state to any other state in a finite number of sampling periods. (Thus, the concept of controllability is concerned with the existence of a control vector that can cause the system's state to reach some arbitrary state.)

Observability is concerned with the problem of determining the state of a dynamic system from observations of the output and control vectors in a finite number of sampling periods. A system is said to be observable if, with the system in state $\mathbf{x}(0)$, it is possible to determine this state from the observation of the output and control vectors over a finite number of sampling periods.

The concepts of controllability and observability were introduced by R. E. Kalman. They play an important role in the optimal control of multivariable systems. In fact, the conditions of controllability and observability may govern the existence of a complete solution to an optimal control problem.

In the second part of this chapter we discuss the pole placement design method and state observers. Note that the concept of controllability is the basis for the solutions of the pole placement problem and the concept of observability plays an important role for the design of state observers. The design method based on pole placement coupled with state observers is one of the fundamental design methods available to control engineers. If the system is completely state controllable, then